# Thinking Dynamically

7-10-23

A (discrete time) dynamical system is a function from a set $X$ to itself.

= "$f: X \to X$" =

idea: connections between dynamical systems and properties of numbers

→ two things

→ a space (our set $X$)

→ a transformation of the space (function)

stuff that gets plugged into function

$$f: X \to X$$

function    stuff that comes out of function

# Examples

$X =$ all real numbers (aka the number line)

aka $\boxed{\mathbb{R}}$

$f: \mathbb{R} \to \mathbb{R}$

$f(x) = \frac{1}{2}x$

we want to iterate this function!

orbit  if I know where I am now, where am I in the future?

$x_0 = 6$

$x_1 = f(x_0) = \frac{1}{2} \cdot 6 = 3$

$x_2 = f(x_1) = \frac{1}{2} \cdot 3 = \frac{3}{2}$

$x_3 = f(x_2) = \frac{3}{4}$

"evolution rule"

sequence converges to zero!
(a geometric sequence)

## Notation

$f^k(x) = k^{\underline{th}}$ step in the orbit

**WARNING**

$$f^k(x) \neq (f(x))^k$$

it is

$$\underbrace{f(f(\ldots f(x))}_{k\text{-times}}$$

The (forward) orbit of $x$ is the set of

$$\{x, f(x), f^2(x), \ldots\} = \mathcal{O}(x)$$

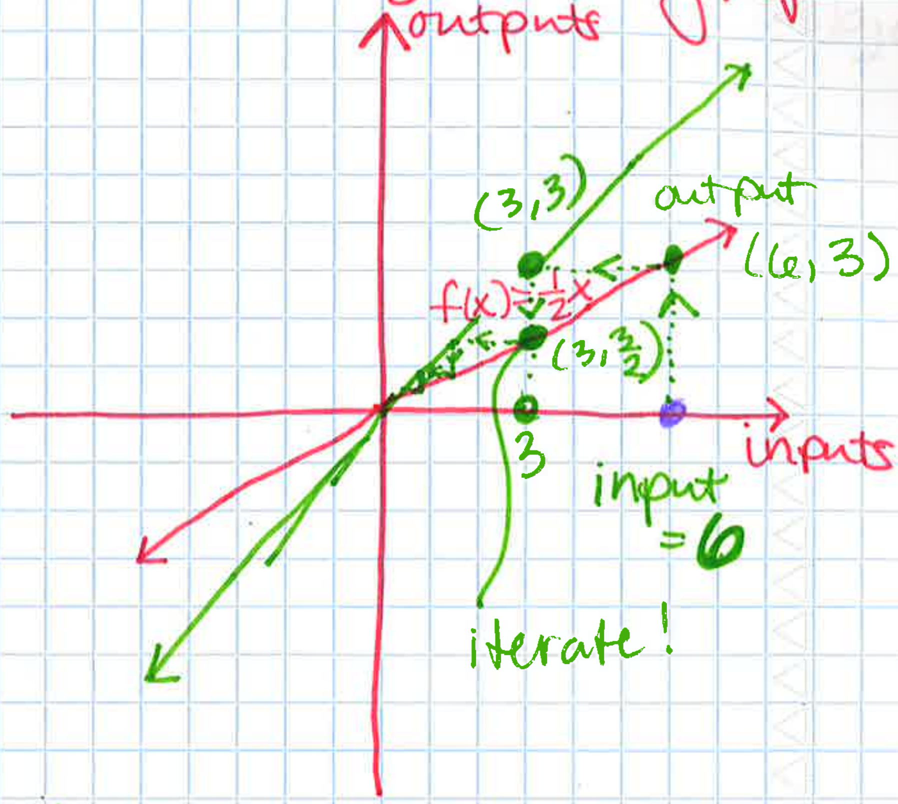
not all dynamical systems are invertible

$f^{-1}(x) = 2x$ (in this example

more on this later!)

# Drawing pictures of the iterates

$f: \mathbb{R} \to \mathbb{R}$ gives a graph




now change y-value into x-value to iterate

to trace iteration/orbit of system

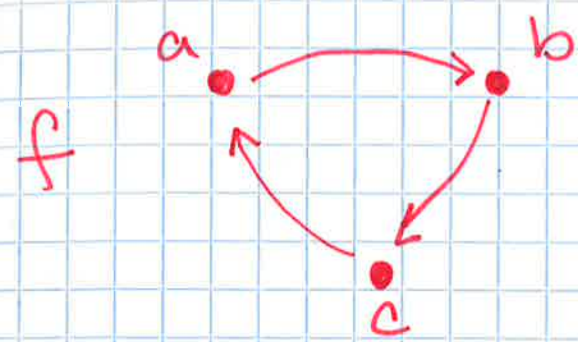
follow path from point on graph toward the line $f(x)=x$, move toward the graph in the output (y) direction

{Repeat!}

# Example 2

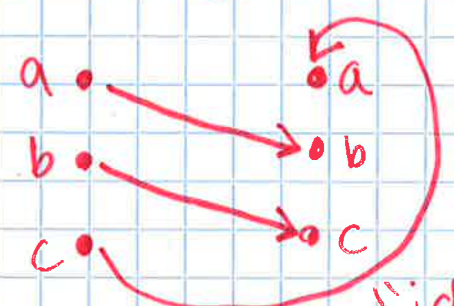
$X$ = a finite set (here the set of 3 objects)

$f: X \to X$ defined by




$f(a) = b$, $f(b) = c$, $f(c) = a$

or like




[notation]
$f^3 = id$
"identity" transformation

| | |
|---|---|
| $f^2(a) = c$ | $f^3(a) = a$ |
| $f^2(b) = a$ | $f^3(b) = b$ |
| $f^2(c) = b$ | $f^3(c) = c$ |

observation:

for K a multiple of 3

$f^k = id$

$f^7 = f$, $f^{14} = f^2$

only three transformations in this system

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<u>**Def**</u> A point $p$ is called <u>periodic</u> if $f^n(p) = p$ for some positive integer $n$.

The smallest such integer is called <u>the period</u>.

Q: on example 1) are there any periodic points in this system?

yes, 0, and its period is 1.

def]
A point is fixed if $f(p)=p$.

all of the periodic points can be detected by looking at the graph (more on this later)

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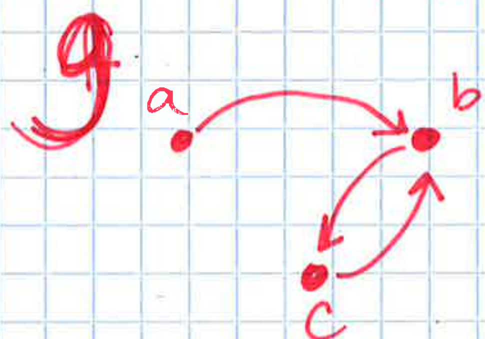
ex 3] $X = \{a, b, c\}$

$g: X \to X$




b and c are periodic of period 2.

Consider the orbit of a]

not periodic but almost.

def] a point p is preperiodic if $f^m(p) = f^n(p)$ for some $m \neq n$.
(ie eventually periodic)

X = the circle

= the boundary of a disk
(think unit circle)
all of the points distance 1 from the origin.

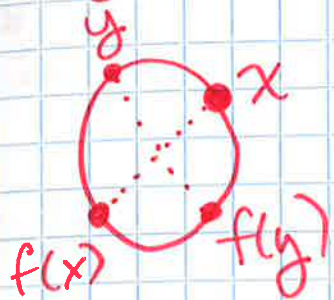
Pick an angle $\theta$ (in radians)


f sends x to a point on the circle rotated counter clockwise from x at an angle $\theta$.

circle rotation with angle $\theta$.

## example (cont)

$\theta = \pi$




every point is periodic with period 2.

[aside: a finite dynamical system is a system whose space/set is finite]

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## Challenge Example ☺

$X$ = the space of finite subsets of rational numbers w/ at least two elements

so points are now sets

ie $\{\frac{1}{3}, \frac{2}{5}, \frac{4}{7}\} \in X$

"belongs to"

$f\left(\left\{\frac{1}{3}, \frac{2}{5}, \frac{4}{7}\right\}\right)$

defining

$$:= \left(\left\{\frac{1}{3}, \frac{1+2}{3+5}, \frac{2}{5}, \frac{2+4}{5+7}, \frac{4}{7}\right\}\right)$$

"elementary schooler's dream"

½ (reduce before adding across

$$= \left(\left\{\frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{6}{12}, \frac{4}{7}\right\}\right)$$

{Some interesting thoughts & questions}

- start with any pair $\{a, b\}$

→ do you eventually see all rational numbers between a & b?

- what happens to the gaps between succesive elements? (how quickly do they go to zero?)
- what is the denominator size? how quickly do they grow?